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Assessing the Development of Preschoolers' Mathematical Patterning

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The development of patterning strategies during the year prior to formal schooling was studied in 53 children from 2 similar preschools. One preschool implemented a 6-month intervention focusing on repeating and spatial patterns. An interview-based Early Mathematical Patterning Assessment (EMPA) was developed and administered pre- and postintervention, and again following the 1st year of formal schooling. The intervention group outperformed the comparison group across a wide range of patterning tasks at the post- and follow-up assessments. Children from the intervention group demonstrated greater understanding of *unit of repeat* and *spatial structuring*, and most were also able to extend and explain growing patterns 1 year later. In contrast, most of the comparison group treated repeating patterns as alternating items and rarely recognized simple geometrical patterns. The findings indicate a fundamental link between patterning and multiplicative reasoning through the development of composite units.

Key Words: Children's strategies; Early childhood; Patterns, relationships in mathematics; Pre-algebra; Preschool/primary; Program/project assessment; Representations, modeling; Visualization/spatial reasoning

Current research on early mathematical development adopts the view that young children are capable of abstraction and generalization of mathematical ideas. An intuitive awareness of patterning concepts and structural relationships has been found more critical to fundamental mathematics learning than previously considered. Prealgebraic thinking develops through a structural awareness of patterns (Carraher, Schliemann, Brizuela, & Earnest, 2006; Mason, Stephens, & Watson, 2009; Sarama & Clements, 2009). Increasingly, studies of conceptual growth have shifted attention from number and arithmetic processes to focus on early algebraic thinking and mathematical reasoning (Carraher & Schliemann, 2007; Clements & Sarama, 2007a; Davis, 1985; English, 2004; Kaput, 2008).

A recent surge of research interest in algebra in elementary school (Blanton & Kaput, 2002, 2003; Carraher & Schliemann, 2007; Kaput, 2008; Moss & Beatty, 2006a, 2006b; Warren & Cooper, 2008), and preschool (Papic, Mulligan, & Mitchelmore, 2009; Perry & Dockett, 2008; Swoboda & Tatsis, 2009; Tzekaki & Ikonomou, 2009; Waters, 2004) is beginning to provide the necessary research base. New mathematics programs (for children aged 4 to 8 years) and aligned professional development initiatives now extend beyond basic numeracy to emphasize aspects of patterning and structural relationships, including equivalence, growing patterns, and functional thinking (Mulligan & Mitchelmore, 2009; Warren & Cooper, 2005, 2008). Spatial structuring is considered necessary for visualizing and organizing these mathematical structures (Battista, 1999).

Recognizing the structure of a pattern—initially, a simple repetition—is central

to the notion of unit of repeat and the development of composite units (Steffe, 1994). Thus, patterning, even at a primitive level of skip counting, can play an important role in the development of multiplicative reasoning (Mulligan & Mitchelmore, 1997; Nunes, Bryant, Burman, Bell, Evans, & Hallett, 2009). The early patterning experiences of young children often involve simple repetition using one variable (e.g., blue, red, blue, red) (Papic, 2007). This may account for difficulties that older students face in recognizing and generalizing patterns and relationships. Often, the teaching of patterning skills focuses on additive thinking, rather than on multiplicative thinking, which is necessary for developing composite units in complex repetitions, and on constructing growing patterns and functional relationships (Mulligan & Mitchelmore, 1997). This article describes the first phase of a design study: the development of an assessment instrument to evaluate the influence of an early intervention program on preschoolers' mathematical patterning.

A reconceptualization of early childhood mathematics curriculum in Australia and internationally has influenced current research agendas with a common goal to place more emphasis on mathematics in early learning (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2010; Clements, 2007). But algebraic reasoning has not previously been central to mathematics syllabi or early mathematics curriculum. The forthcoming Australian National Curriculum (ACARA, 2010) promotes early algebra in the elementary grades. "An algebraic perspective can enrich the teaching of number . . . and the integration of number and algebra, especially representations of relationships, can give more meaning to the study of algebra in the secondary years. This combination incorporates pattern and/or structure and includes functions, sets and logic" (ACARA, 2009, p. 6). However, there is currently insufficient research, or impetus from professionals, to support an early childhood mathematics curriculum that integrates a specific learning model of early algebra (Sarama & Clements, 2009).

At a more pragmatic level, policymakers and educators aim to narrow the "gap" between high and low achievement by providing effective and accessible mathematics programs (Clements & Sarama, 2009; Perry & Dockett, 2008). Research and development of early intervention strategies focus not only on prevention of later learning difficulties but also on promoting opportunities for rich mathematical development prior to formal schooling (Clarke, Clarke, & Cheeseman, 2006; Clements & Sarama, 2007b; Doig, McCrae, & Rowe, 2003; Sarama & Clements, 2009; van Nes & de Lange, 2007). The quality, scope, and depth of both the teaching and assessment of early mathematics are regarded as critical to future success in the subject (Aubrey, Dahl, & Godfrey, 2006; Davis, 1985; Doig, 2005; Wright, 2003; Young-Loveridge, Peters, & Carr, 1998).

THEORETICAL FRAMEWORK

Pattern

We understand the term *pattern* to mean any replicable regularity. Patterns occur in three contexts that are of particular significance in mathematics:

1. Within a single object, in which some of its components are consistently related;
2. Within an ordered set of objects, in which there is a consistent relation between each component and the next; and
3. Between two ordered set of objects, in which the corresponding elements are paired in some way.

In early childhood mathematics (i.e., up to Grade 2), children experience all three types of pattern. At the preschool level, only the first two types are considered appropriate. Typically, they are represented by the following types of patterns (Papic & Mulligan, 2007):

- Spatial structure patterns are invariant relations between various features of geometrical shapes. Examples of shapes are triangles, squares, blocks, arrays, and grids; examples of features are the number, size, collinearity, and spacing of the elements of these shapes.
- Repeating patterns have “a cyclic structure that can be generated by the repeated application of a smaller portion of the pattern” (Liljedahl, 2004, pp. 26–27). The simplest example is the alternating pattern symbolized by ABAB . . . . The generating portion (in this case, AB) will be referenced in this article as the unit of repeat (Threlfall, 1999).

An additional type of pattern has received much attention in the early algebra curriculum and related research (Kaput, 2008) and has been shown to be appropriate for the early years of formal schooling (Warren & Cooper, 2008):

- Growing patterns consist of sequences of elements that increase (or decrease) systematically. A simple example is the sequence of even numbers 2, 4, 6, ... The elements of a growing pattern may themselves be spatial patterns.[1]

Growing patterns may be regarded as occurring within an ordered set of objects. For example, the even numbers increase by 2. Alternatively, growing patterns may be seen as involving a relationship between two sets of objects, namely, the sets of objects and their positions in the sequence. From this viewpoint, each even number is twice its position number (2, 4, 6, . . .).

Algebraic Thinking

Algebra may be viewed as a symbolic language that enables one to express relationships and generalizations, usually involving numbers, and to use them to solve problems without the extensive numerical computations that might otherwise be necessary (e.g., when using trial and error). The central idea is that of a generalization, that is, a relationship that holds over an entire class of values, not only in isolated instances (Dörfler, 1991). From this perspective, finding and using generalizations may be considered *algebraic thinking*.

[1]For example, in the growing pattern 1, 3, 6 in Table 4, each element is a triangular spatial pattern.

A particular type of algebraic thinking is *functional thinking*, which occurs when an individual perceives a generalization that relates two sets of objects. Such relations permeate mathematics in various guises (e.g., functions, correlations, and transformations). Treating a growing pattern in a sequence of elements as a relation between each element and its position in the sequence is an elementary example of functional thinking.

Every pattern is a type of generalization in that it involves a relationship that is "everywhere the same." Working with patterns therefore involves algebraic thinking. At the preschool level, this perspective is most easily justified in the case of repeating patterns for which the consistent relation between successive elements (the units of repeat are identical) is as simple as can be imagined. Generalization also occurs, for example, when a child recognizes that the structure of the unit of repeat in patterns constructed from different materials is "the same." Even the exploration of spatial structural patterns may be regarded as algebraic thinking, because the aim is to find consistent relations within specific categories of geometrical shape. Because patterning activities in preschool often do not involve numbers, we shall refer to the finding of generalizations at this level as *prealgebraic thinking*.

Mason (1996; Mason, Drury, & Bills, 2007; Mason, Graham, & Johnston-Wilder; 2005; Mason, Stephens, & Watson, 2009), like many other contemporary researchers, has argued for a focus on generalization in the elementary years. According to Lins and Kaput (2004), Mason's perspective is representative of the viewpoint that "students come to school with natural powers of generalisation and abilities to express generality, and that the development of algebraic reasoning is, in large part, a matter of tapping into those naturally occurring capacities for didactic purposes" (Lins & Kaput, 2004, p. 54). However, Carraher et al. (2006, p. 92) warn that deciding whether algebra concepts are able to be understood by young children requires "empirical studies with young students who have had access to activities and challenges that involve algebraic reasoning and algebraic representation." Young children have indeed been observed developing skills in argumentation (Dockett & Perry, 2001) and generalization (Blanton & Kaput, 2004). However, the research base for establishing the relative importance of algebraic thinking across content domains in early mathematics remains undeveloped.

Research on Mathematical Patterning in Early Childhood

Research on early mathematics learning has often been restricted to an analysis of children's developmental levels of single concepts such as counting, but has not provided insight into common underlying processes, such as patterning and spatial structuring, that are salient to many mathematical concepts (Clements & Sarama, 2009; Ginsburg, Lee, & Boyd, 2008; Mulligan & Mitchelmore, 2009; Perry & Dockett, 2008). Children's patterning knowledge has also been found to underpin the development of analogical and inductive reasoning (English, 2004).

The importance of spatial structuring in mathematics education has long been recognized in early mathematics education (Battista, 1999; Bobis, 1996). Some studies indicate that spatial structuring ability directly supports the development

of number sense (van Nes & de Lange, 2007) and structural relationships across mathematical content domains (Mulligan, Mitchelmore, Kemp, Marston, & Highfield, 2008; Mulligan, Prescott, & Mitchelmore, 2004). However, gaining deeper insight into early algebraic reasoning may require researchers to focus on the relationship between spatial structuring, generally, and the structural development of patterning concepts.

Several studies have found that young children apply patterning skills in a wide variety of situations, including simple repetition (Young-Loveridge, Peters, & Carr, 1998), part–whole thinking (Hunting, 2003; Lamon, 1996; Young-Loveridge, 2002), recognizing spatial and geometric patterns (Feeney & Stiles, 1996), subitizing[2] (Bobis, 1996; van Nes, 2009), and counting using calculators (Groves & Stacey, 1998).

The role of patterning has been explored in early childhood programs designed to enhance mathematical development generally, for example, in The Berkeley Math Readiness Project (Klein & Starkey, 2003), Big Math for Little Kids (Ginsburg, 2002), Building Blocks (Clements & Sarama, 2007b), the Mathematics and Neurosciences (MENS) research project (van Nes & de Lange, 2007), and the Pattern and Structure Mathematics Awareness Program (PASMAP) (Mulligan et al., 2008; Mulligan, Mitchelmore, & Prescott, 2006). The PASMAP researchers have documented structural features of children's early mathematical development across a wide range of number, measurement, and space concepts, and they assert that children's Awareness of Mathematical Pattern and Structure (AMPS) in the first 3 years of schooling can be regarded as a general cognitive characteristic (Mulligan & Mitchelmore, 2009).

Contemporary studies of children's early algebraic thinking often focus on the age range of 6 years to 8 years. For example, Warren and Cooper (2005, 2008) have shown that 8-year-old children in Grade 2 can successfully learn functional thinking through the analysis of growing patterns. There have been relatively few studies of patterning in preschool settings. One recent observational study (Waters, 2004) found that Australian preschoolers initiate and talk about their own patterns, ranging from simple repetition to geometric forms. Other studies have focused on children's patterning language (Swoboda & Tatsis, 2009) and how they represent spatial relationships through patterning (Tzekaki & Ikonomou, 2009). Few studies have examined the development of patterning in the context of technological tool usage (Highfield & Mulligan, 2007; Moss & Beatty, 2006a, 2006b), but the potential for further development is highlighted by the Building Blocks project (Clements & Sarama, 2007b).

However, there remain many unanswered questions about how and when early algebraic thinking develops in the years prior to formal schooling (Carraher et al., 2006; Sarama & Clements, 2009). Further research is needed to describe explicitly how patterning can lead to the development of mathematical concepts and prealgebraic reasoning in young children.

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[2]Subitizing is the rapid, accurate, and confident judgment of the numerosity of a small collection of items.

The Present Study

This study was designed to develop and evaluate a research-based preschool numeracy program focused on patterning. Because the program was conceived as a specific learning model (Clements, 2007), our goal was to develop an assessment interview, a corresponding instructional framework based on patterning as a core learning trajectory, and a teacher professional development program. Three key research questions are addressed:

1. What are the characteristics of mathematical patterning that young children develop prior to formal schooling?
2. What influence does an intervention that promotes mathematical patterning have on the advancement and complexity of preschoolers' patterning concepts and skills?
3. Is the influence of such an intervention maintained after children's 1st year of formal schooling?

METHOD

Setting and Participants

The research was conducted at two privately owned preschools in the southwestern region of Sydney, Australia. One preschool was assigned as the experimental intervention preschool (IP) and the other as the comparison nonintervention preschool (NP). Both preschools had equal enrollments (38 children in each school per day), comparable staffing levels and resources, and similar approaches to curriculum. During the intervention, there were no changes in the teaching staff at either preschool. Approximately 80% of the children were from non-English-speaking backgrounds, but all were competent in English.

The 53 participants were from 3 years 9 months to 5 years of age. All participants were retained for the duration of the intervention period and spent a minimum of 6 hours per day at preschool for at least 2 days per week. Eighteen children were unavailable for the second assessment because many families had commenced their holidays earlier than expected. For the third assessment, all of the initial 53 participants were tracked. Apart from 4 who had relocated to another state or overseas, all remaining 49 were tracked to Kindergarten classes in 22 local elementary schools. However, for a variety of reasons (including a measles outbreak in the area, lack of parental consent, and children commencing summer holidays earlier than expected), only 32 were available for the third assessment.

Table 1 shows the number of IP and NP children included at each assessment point. Note that attrition was more severe among the NP than the IP children, especially at the third assessment. Because the children were enrolled in so many schools, there was no obvious explanation for this result. To assess any possible effects for differential attrition, mean total scores (percentage correct) at the first assessment were calculated for the children who were available at each assessment

(see Table 1). The mean scores for the children who were assessed at the second and third assessments are all within one standard error for the means of random samples of the given size taken from the original sample. This finding shows that the children in the second and third samples were representative of the original sample, both in the IP group and the NP group.

Design

The study was formulated as the first iteration of a design study (Gravemeijer & van Eerde, 2009), incorporating several foundational phases: (a) determining and describing critical patterning knowledge, (b) developing an assessment instrument to describe levels of conceptual development and strategy use in patterning tasks, (c) developing and refining an intervention program based on an instructional framework, (d) evaluating the impact of the intervention on advancing mathematical development, and (e) building a "patterning" learning trajectory aligned with an instructional framework. In this article, we report on the first two phases of the study.

The researcher (first author) assessed children's patterning knowledge at the beginning and end of the preschool year and again at the end of their 1st year of formal schooling. Table 1 summarizes the timing of assessments. An independent numeracy assessment measure, The Schedule for Early Number Assessment 1 (SENA 1) (NSW Department of Education & Training, 2001) was also administered at the third assessment. Following the preassessment, and over the ensuing 6-month period, the researcher collaborated with the IP teachers to develop and implement an instructional framework incorporating *structured, scaffolded, pattern-eliciting tasks* that reflected some of the EMPA tasks and the children's patterning strategies demonstrated at the first assessment. Another instructional strategy included the "patternizing" of the regular preschool program as well as the naturalistic observation of children applying patterning in unstructured play.

The researcher worked initially with the IP teachers for 4 weekly half-day periods

Table 1
Numbers of Participants at Each Assessment and Their Mean Scores (*Percent Correct*) *on Assessment 1*

			IP		NP	
Year	Month(s)	Assessment	Number	Mean (*SD*)	Number	Mean (*SD*)
Preschool	June	1	27	26 (20)	26	34 (20)
Preschool	December	2	19	30 (21)	16	32 (19)
Kindergarten	November/ December	3	20	30 (19)	12	37 (21)

Note. IP stands for the Intervention Preschool group; NP for the Nonintervention Preschool (comparison) group.

to develop their pedagogical content knowledge about patterning and to plan the intervention. During the implementation, she visited for 1 day each week assuming the role of *participant observer*, during which time she assisted in teaching, observing children's behavior, and collecting program evaluation data. Data collection extended across the 5-day weekly period and included systematic records made by the preschool teacher. Close monitoring of the children's patterning was obtained as naturalistically as possible—they were accustomed to teachers making observation notes and digital recordings of classroom activities.

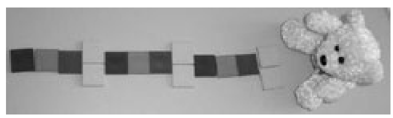
The Assessment Instrument

An Early Mathematical Patterning Assessment (EMPA) was developed, trialed, and refined prior to commencement of the intervention. The Interview process and tasks for the first assessment were trialed to ensure that the interviews were well prepared and consistent and that instructions were clear and resources were appropriate. Trialing of all tasks was conducted with 8 preschool children. Four of the children were from the intervention preschool and 4 were from the nonintervention preschool. The children were not part of the sample in the study, because they were not enrolling in kindergarten the following year. The children were, however, similar in ages to those in the main study. A variety of tasks was devised to assess children's facility with repeating patterns, spatial structure, and growing patterns (see Tables 2–4). The first and second assessments consisted of identical sets of Repeating Pattern tasks (8 Towers tasks, 1 Borders task, 6 Hopscotch tasks, and 1 Numbers task) and Spatial Structure tasks (2 Arrays tasks, 1 Grids task, 4 Subitizing tasks, and 2 Triangles 1 tasks). For the third assessment, the Repeating Pattern tasks (6 Towers tasks, 3 Borders tasks, 1 Hopscotch task, and 2 Numbers tasks) were increased in complexity and 4 Growing Pattern tasks were added (2 Squares tasks and 2 Triangles 2 tasks). To allow time for the additional tasks, Spatial Structure tasks were omitted from the third assessment.

During the first phase of the research, we reviewed research on early mathematics learning and the development of algebraic reasoning generally. Most important, we looked for evidence of the relationships between children's patterning skills and more general numeracy development. Our early review encompassed research on counting, subitizing, partitioning, spatial structuring, and geometric development, as well as studies of multiplicative reasoning. Earlier studies of the relationship between multiple counting and patterning, the role of composite unit and unitizing influenced the scope and direction of the inquiry (Mulligan & Mitchelmore, 1997; Outhred & Mitchelmore, 2000; Steffe, 1994). The aim was to develop initially an assessment of a broad range of problems or tasks on patterning that preschoolers may be able to solve.

The tasks were based on items in published research studies. The Tower tasks were inspired by Maher's longitudinal study (2002), in which children solved combinatorial problems using towers made with colored blocks. The Spatial Structure tasks were informed by studies on visual memory and pattern recognition

Table 2
Categories, Descriptors, and Exemplars of EMPA Repeating Pattern Tasks

Category	Descriptors	Exemplar task and typical response
Towers	Copy, continue, represent, and identify screened element, simple (AB, ABC, ABBC) repetitions using blocks and by drawing. Design and draw tower patterns from memory.	Draw a 6-block AB repetition from memory. 
Borders	Complete border patterns using cut-out tiles. Identify whether a border pattern has a clear start or finish. Justify why a border pattern cannot be completed.	Use the tiles to complete the border pattern. 
Hopscotch	Copy, complete, and represent a hopscotch pattern, both directly and rotated 180˚ using square tiles and by drawing. Copy and draw hopscotch pattern from memory. Design own hopscotch pattern.	Design own hopscotch pattern. 
Numbers	Identify next numeral and color in pattern of two numerals using two colors. Identify the element of the number pattern.	Identify the next numeral and color in the pattern [colors alternate black and red]. 1 2 1 2 1 2

(Bobis, 1996; Mulligan, Prescott, & Mitchelmore, 2003) and on pattern and structure in various contexts (Battista, 1999; Mulligan, Prescott, & Mitchelmore, 2004; Outhred & Mitchelmore, 2000). The ability to subitize is considered fundamental in developing visual memory and pattern recognition (Bobis, 1996; Hunting, 2003; Wright, 1994). The subitizing tasks were adapted from the Schedule for Number Assessment 1 (SENA 1) (NSW Department of Education & Training, 2001).

The tasks were presented in a variety of formats. In several items, children were simply shown a pattern (such as a completed tower of blocks) and asked to copy it. In some of these, the pattern was removed before children copied it, or part of the pattern was covered. Many tasks required children to extend a given pattern. Other tasks were more open-ended, for example, when children were asked to make a

Table 3
Categories, Descriptors and Exemplars of EMPA Spatial Structure Tasks

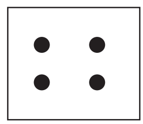
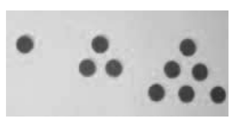
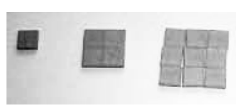
Category	Descriptors	Exemplar task and typical response
Arrays	Copy array patterns made with 4, 5, or 6 counters, using counters and by drawing.	Draw a 5-dot array pattern. 
Grids	Copy rectangular grid patterns made with 2, 3, and 4 squares by drawing.	Copy a rectangular grid pattern by drawing. 
Subitizing	Identify number of dots in regular and irregular patterns and within grids (3–6 dots). Identify number of blocks in staircase block patterns (3 and 5 blocks).	Identify the number of dots after viewing the pattern for 3 seconds. 
Triangles 1	Copy triangular dot patterns made with 3 and 6 counters, using counters and by drawing.	Draw a triangular dot pattern. 

Table 4
Categories, Descriptors, and Exemplars of EMPA Growing Pattern Tasks

Category	Descriptors	Exemplar task
Triangles 2	Continue triangular pattern of dots and explain response.	Continue this pattern: 
Squares	Continue square pattern of blocks and explain response.	Continue this pattern: 

hopscotch pattern. A few tasks required a simple verbal response, for example, indicating the number of dots in a pattern. Tasks often required children to explain their responses, and in other tasks the interviewer prompted children to explain any unexpected or unusual responses.

Data Collection and Analysis

Administration of the EMPA followed a semistructured individual interview protocol, with the task order and procedures at each assessment remaining constant. The interviews were conducted by the first author in a small room adjacent to children's classrooms in two 20-minute sessions on successive mornings. Children were provided with paper, colored pencils, and a variety of other materials. All interviews were audio recorded. A random sample of 20% of the interviews was also video recorded to facilitate the classification of children's responses.[3]

The analysis of children's EMPA responses proceeded as follows. First, children's EMPA responses were coded for accuracy. For the Repeating Patterning tasks, it was expected that preschool children would extend the given patterns by replicating the most obvious unit of repeat. For example, although theoretically the pattern 1 2 1 2 1 2 could be continued in any number of ways, the most obvious method is to repeat the 1 2 unit. In assessing children's responses, the crucial criterion was whether children explained their continuation by reference to some pattern implicit in the given stimulus. Although some students used the unit of repeat in a more sophisticated manner than others, all responses that used a unit of repeat coherently were considered as correct. Similar considerations applied to the Growing Pattern tasks.

Second, their various strategies on each task were described, making use of children's drawn representations and their recorded explanations, and supplemented by the video recordings, observation notes, and some still photographs. These strategies were then classified into a small number of increasing levels of sophistication, focusing on the structure of the representation and the use of a unit of repeat. Minor inaccuracies in drawing or positioning were ignored. An independent coder recoded all children's responses, yielding an intercoder reliability of 89%.

The Schedule for Early Number Assessment 1 (SENA 1) (NSW Department of Education & Training, 2001) was administered at the third assessment. SENA 1, which has been developed over several years and is used in the state-wide assessment program, measures children's level of number knowledge and arithmetical strategies on 56 items on Numeral Identification, Forward and Backward Number Word Sequence, Counting, Addition, Subtraction, Multiplication, and Division. It was administered after the EMPA as an individual interview in a separate 20-minute period.

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[3]It was determined that 20% was sufficient to show the various strategies that children used to solve tasks, because the data set also included photos and audio recordings. The video data assisted with classifying the responses into various categories.

The Intervention

The intervention functioned primarily as an enrichment program through the addition of problem-based patterning tasks to the existing program. It comprised the following two components: *structured, individualized, pattern-eliciting tasks* and *"patternizing" the regular preschool program*. The NP group participated in its regular preschool program, with no explicit focus or instruction on patterning.

Structured, individualized, pattern-eliciting tasks were based on the EMPA Towers, Subitizing, and Hopscotch tasks. The Towers tasks were the main focus of the structured individual and small-group work. For example, children drew from memory block towers with two, three, or four repetitions of a three-color pattern (e.g., RBG); found a missing block (e.g., in RBGR_G); identified errors (e.g., in RBGRBGBRG); and created new patterns using different materials and in different orientations. The tasks were based on instructional frameworks (see Table 5 for Tower tasks framework) that were constructed from children's responses at the first assessment. Through these tasks, children were encouraged to identify similarities and differences within and between patterns, to identify the unit of repeat and the number of repetitions wherever appropriate, and to explain their thinking.

Children were placed on the Framework based on their EMPA responses. Children progressed through all the tasks at their level. There was flexibility in the order of the tasks to ensure that each child's learning was being scaffolded. A number of tasks were selected for each session. During group time[4] each day, usually in the morning, one teacher would implement the planned framework tasks with individual children or small groups of children. Children, who were on the same level, completing similar tasks, were grouped together. A combination of individual and group time was allocated. This varied for individuals and depended on the level at which the child functioned, the number of children at the same level on that particular day, and the concentration span of the child or children. Children progressed to the next level if they showed competency at the current level. This was determined by the teachers with the assistance of the researcher after reviewing the children's individual progress and strategy use.

It was essential that all children were exposed to the same number of tasks using the same procedures. Therefore sessions were programmed on a fortnightly basis to ensure that children who were present only 2 days per week would receive the same amount of time on the structured tasks as children enrolled 5 days per week. One 30-minute session was scheduled each fortnight for each child for a period of 18 weeks. These were often broken into two smaller 15-minute sessions, depending on the children's attention span. Twelve weeks were devoted to Tower tasks, 4 weeks to Spatial pattern tasks, and 2 weeks to Hopscotch tasks.

"Patternizing" the regular preschool program emerged as the teachers realized how much children were learning from the initial patterning activities. The first author worked collaboratively with teachers to scaffold rich patterning experiences

[4]This is the time of day for which teachers plan group activities around themes or interests or around concepts or skills they wish to develop.

Table 5
Instructional Framework for Tower Tasks

Level 1: Prestructural stage	Level 2: Emergent stage
1. Copying 2-block tower with blocks	1. Copying 4-block ABAB tower (2 colors × 2)
2. Copying 2 block tower by (a) constructing with colored tiles and (b) drawing with textas[5]	2. Drawing 4-block ABAB tower (2 colors × 2) by copying using textas
3. Designing own 2-block tower	3. Designing own 4-block ABAB tower (2 colors × 2)
4. Drawing 2-block tower from memory	4. Drawing 4-block ABAB tower (2 colors × 2) from memory
	5. Continuing tower pattern to make 6-block tower (2 colors × 3)

Level 3: Structural stage	Level 4: Advanced structural stage
1. Copying 6-block tower ABABAB (2 colors × 3)	1. Copying block towers with 3 color repetitions (ABC) × 2, × 3, and × 4
2. Drawing 6-block tower ABABAB (2 colors × 3) by copying using textas	2. Drawing by copying block towers with 3 color repetitions (ABC) × 2, × 3, and × 4 using textas
3. Designing own 6-block tower ABABAB (2 colors × 3)	3. Designing own block towers with 3 color repetitions (ABC) × 2, × 3, and × 4
4. Drawing 6-block tower ABABAB (2 colors × 3) from memory	4. Drawing from memory block towers with 3 color repetitions (ABC) × 2, × 3, and × 4 using textas
5. Continuing tower pattern to make 8-block tower (2 colors × 4)	5. Continuing tower pattern (e.g., RBGRBG_ _ _ YBOYBOYBO_ _ _)
6. Finding the missing block or error (e.g., RBRBBR RBRB_B)	6. Finding the missing block or error (e.g., RBGRBGRBR RBGRBG_BG)
	7. Continuing various complex single variable patterns (e.g., ABBABBABB _ _ _ ABBCABBCABBC_ _ _ _)
	8. Copying various complex, single-variable repetitions from memory
	9. Designing various complex, single-variable repetitions
	10. Drawing various complex, single-variable repetitions from memory using textas

[5]A texta is a felt-tipped pen, usually colored and wide-tipped for use by children.

and explorations within their regular curriculum. For example, as part of children's investigation of reptiles, the teacher discussed the patterns in the serpent's body in the text *Kun-Man-Gur, The Rainbow Serpent* (Cowan & Bancroft, 2000). Children made their own serpents, designing repeating patterns on the snake's body. This pedagogical strategy replicated, to some extent, the approach taken by Blanton and Kaput (2003) when they "algebrafied" school mathematics.

As well as observing children during these structured activities, the first author also observed children's free play for 1 hour per visit and recorded instances in which children initiated pattern-making activities, such as simple or complex repetition. As part of their regular observation of children, the staff also kept records of spontaneous instances of patterning they observed. As with Fox's (2005) study, systematic observations were made to ascertain whether children initiated experiences that incorporated various patterns. In this context, the teachers' role was not to scaffold the learning: It was purely to observe and document instances in which children used patterns in their free play. Teachers provided various resources that may have encouraged patterning and ensured that there were enough resources available for children to design different types of repetitions. If children shared what they were doing with the teacher and acknowledged that they had created a pattern, then the teacher would question the children further to encourage them to justify their pattern and to extend the children's existing knowledge. Teachers' documentation provided evidence of the growth in their pedagogical content knowledge as well as useful information for follow-up experiences.

QUANTITATIVE RESULTS

In this section, we compare the accuracy with which children in the intervention preschool (IP) and the nonintervention preschool (NP) solved the various patterning tasks included in the EMPA. Accuracy was measured by the mean number of correct responses, expressed as a percentage of the total number of tasks in that category. No statistical tests were applied because the two samples were not randomly selected.

Table 6 shows children's accuracy at Assessment 1 (at the start of the intervention) and Assessment 2 (at the end of the intervention). The two groups of children performed at very similar accuracy levels at Assessment 1 with the NP children showing a slight superiority. However, at the end of the intervention period, the IP group outperformed the NP group on all tasks, in some cases by a large margin. The contrast was particularly evident for the Numbers, Grids, Subitizing, and Triangles 1 tasks on which the IP children improved substantially, but the NP children showed little or no improvement. Note that, of these four tasks, only the Subitizing tasks resembled any of the pattern-eliciting tasks used extensively in the intervention.

Table 7 presents the results of Assessment 3. Recall that the Repeating Patterns tasks were similar to (but more complex than) the corresponding tasks used in the first two assessments, but that the Growing Patterns items were completely new. The results indicate that the IP children continued to show higher performance than

Table 6
Percent Correct Responses at Assessments 1 and 2, by Preschool

Task	Assessment 1		Assessment 2	
	IP	NP	IP	NP
Repeating patterns				
Towers	34	47	85	73
Borders	74	81	100	88
Hopscotch	16	28	55	45
Numbers	11	19	58	19
Spatial structure patterns				
Arrays	47	42	79	72
Blocks	47	46		
Grids	33	27	79	25
Subitizing	15	20	58	16
Triangles 1	7	8	50	13

the NP children on the Repeating Patterns tasks and that almost half of them could extend one or both of the Triangles 2 and Squares patterns and explain their responses.

On the Schedule for Early Number Assessment 1 (SENA 1), administered at about the same time as Assessment 3, the IP children scored considerably higher on average (82%) than the NP children (63%). Table 8 provides data on some selected tasks. These data suggest that, 12 months after the intervention, the IP children's counting and arithmetic skills were considerably more advanced than those of the NP children.

AN ANALYSIS OF CHILDREN'S PATTERNING STRATEGIES

This section presents an analysis of children's patterning strategies as evidenced by their responses to the Repeating Patterns, Spatial Structure, and Growing Patterns tasks. For each of these three types of tasks, we list four or five categories of response in increasing order of sophistication. For each of these categories, we first describe, illustrate, and quantify children's responses and then compare the portions of IP and NP children who provided such responses after the intervention.

Repeating Patterns

An analysis of both correct and incorrect solution strategies for all Repeating Pattern tasks showed that they fell into the following five main categories.

Random arrangement. At Assessment 1, children frequently placed items in a

Table 7
Percent Correct Responses at Assessment 3, by Preschool

Task	IP	NP
Repeating pattern tasks		
Towers	93	47
Borders	53	22
Hopscotch	65	8
Numbers	83	17
Growing pattern tasks		
Triangles 2	38	0
Squares	48	0

Table 8
Percent Correct Answers to Selected SENA 1 Tasks, by Preschool

Task	IP	NP
Recognizes numerals to 100	60	25
Counts to 100 and states the next number	60	8
Counts backward from 30 and states the previous number	50	8
Recognizes number of dots in 8- and 9-dot domino pattern without counting	90	33
Counts on to solve addition and subtraction word problems	55	25

random fashion when attempting to copy or construct repeating patterns. For example, when attempting to copy the hopscotch pattern shown in Table 2, children would scatter tiles around the area in front of the teddy bear without paying any attention to the alignment of the rows and columns or even the number of tiles. Prior to the intervention, 23% of all the children represented the hopscotch pattern in this way. After the intervention, this strategy was not observed among IP children and among only 13% of the NP children; by the follow-up assessment, the strategy was no longer observed.

Direct comparison. Children who used this strategy copied a pattern by matching one item at a time. For example, in Figure 1 the child is proceeding to construct a copy (left side) by direct comparison with the model (right side). Similarly, in the Hopscotch tasks in Table 2, many children traced around the individual tiles to ensure that their picture matched the size of the model.

A large number of the children used this strategy. For example, before the intervention, 28% of the entire sample used direct comparison on at least three of the five Towers tasks that required them to copy or draw a tower. This strategy continued to be the dominant strategy on these tasks for 19% of the NP children at Assessment

2 and 17% at Assessment 3. In contrast, this strategy was not used by any of the IP children on any task at Assessments 2 or 3.

Alternation. Children from both groups often focused on successive items regardless of the complexity of the unit of repeat. For example, when copying a 6-block tower made of alternating red and blue blocks, their actions and comments indicated that they focused exclusively on the sequence of individual colors (red, then blue, then red) rather than on the unit of repeat (red–blue).

The alternation strategy proved successful with simple repetitions, but with an increase in task complexity (e.g., an ABBC repetition), it became ineffective. Its shortcomings were particularly evident in one task in which children were asked to recreate a 12-block tower made up of three repetitions of a red–blue–blue–black unit after it had been screened from view. Many of the children constructed a tower of the correct height but used an incorrect number of red, blue, and black blocks. The following excerpt illustrates the thinking of children who followed this strategy:

Child: You tried to trick me by putting two blues, but that's not a pattern.
Interviewer: It's not a pattern?
Child: No, because it can't have two of the same color next to each other.
Interviewer: Why can't it have two same colors next to each other?
Child: Because it can't. You have to have different colors like red, blue, black. Then it's a pattern.

Similarly, in a Border task, children were asked to determine whether the border of a 3 × 4 pattern of squares could be filled using a red–blue–green repetition of squares. Because the border consisted of 10 squares, the task could not be completed without breaking the pattern. However, many children believed that the border could nevertheless be completed satisfactorily. Their thinking is illustrated in the excerpt on the next page:

Figure 1. Towers pattern being copied using direct comparison.

Interviewer: Can I finish the border pattern?
Child: Yes.
Interviewer: How do you know that you can finish the pattern?
Child: Look. Red, blue, green, red, it's finished.
Interviewer: Does it matter if you finish on a red?
Child: No. It can finish on a red, see.

For all the NP children, alternation was their most common strategy at all assessments. However, although many IP children used this strategy at Assessment 1, it was the most common strategy for only 16% of the IP children at Assessment 2 and it was not used at all by IP children at Assessment 3.

Basic unit of repeat. In this strategy, children identified the unit of repeat independently of the number, type, and complexity of items and of attributes such as size, shape, dimension, and orientation. They could then use the unit of repeat to extend the pattern and complete more complex tasks. For example, at Assessment 3, children were given a tower consisting of three red–blue–blue–black repetitions in which the 9th and 10th block were screened. They were given 10 seconds to view the tower; it was removed from view, and then they were required to draw it from memory. Children who were successful on this task typically first identified the unit of repeat and calculated the number of repetitions using the language of multiplication. For example:

Interviewer: How did you know that you had finished making your tower?
Child: I remembered red, blue, blue, black, three times.

The unit of repeat was also used in patterns in the Hopscotch tasks. For example, Figure 2 shows a child's drawings of a vertical hopscotch pattern at the first two assessment points. At Assessment 1, there was no evidence of a unit of repeat. At Assessment 2, however, the child drew the correct unit of repeat (two squares horizontally and two squares vertically) in one continuous movement and then replicated it twice, pausing noticeably after each unit.

Only 8% of the children used this strategy on any task before the intervention. However, 74% of the IP children used this strategy on at least 11 of the 16 Repeating Pattern tasks at Assessment 2 and 85% of them used it on at least 8 of the 12 Repeating Pattern tasks at Assessment 3. By contrast, only 25% of the NP children used this strategy at any time at Assessment 2 or Assessment 3.

Advanced unit of repeat. After the intervention, some IP children had developed their understanding of the unit of repeat to the point at which they could transfer the same pattern to different materials or modes, or reconstruct the pattern in creative ways. Some IP children also demonstrated and expressed simple generalizations about the unit of repeat.

For example, when designing their own hopscotch pattern at Assessment 2, many IP children used a second variable, color, to form another pattern within the unit of repeat. Figure 3 shows a three–two–one unit of repeat; the child who created it

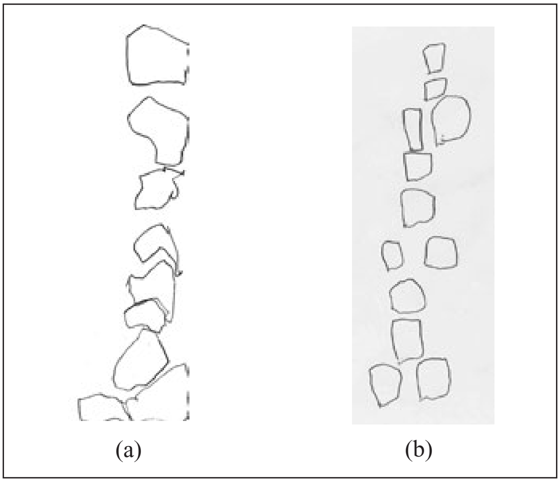

Figure 2. Child's drawing of hopscotch pattern at (a) Assessment 1, and (b) Assessment 2.

verbalized the number pattern and replicated the color pattern exactly.

Another example illustrates the transfer of a pattern structure. An IP child made an ABB tower with blue and yellow blocks and spontaneously compared it with a pattern he had earlier made with flowers:

Child: Look, it's the same pattern as the flowers.

Interviewer: What do you mean, it is the same pattern as the flowers?

Child: Look, this is blue, yellow, yellow, blue, yellow, yellow, blue, yellow, yellow. And the flowers are curved, spiky, spiky, curved, spiky, spiky, curved, spiky, spiky.

Interviewer: How is that the same?

Child: It's the same.

Interviewer: How is it the same?

Child: 'Cause look. There is one curved and one blue, and then there's two spiky and two yellow, that's the same pattern.

The child's insight may represent the emergent generalization of an ABB pattern.

No NP child demonstrated an advanced unit of repeat strategy. For example, although 25% of the NP group designed a hopscotch pattern at Assessment 2 that used a unit of repeat, all their patterns involved simple AB repetitions. In contrast, 63% of the IP group constructed a valid hopscotch pattern that contained a new form of the unit of repeat.

In summary, the results on the Repeating Patterns tasks showed a substantial difference between the IP and NP children. All the IP children developed an understanding of the unit of repeat that they used on several tasks. By contrast, the NP children tended to rely on alternation and only rarely used a unit of repeat.

Spatial Structure

Analysis of children's responses to the Spatial Structure tasks (administered only

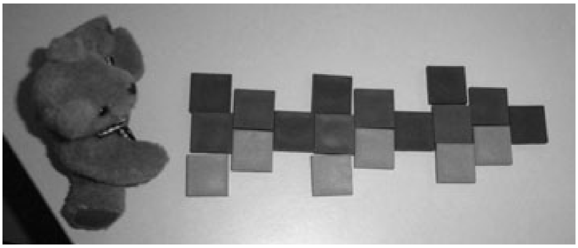

Figure 3. Child's own hopscotch pattern created at Assessment 2.

at Assessments 1 and 2) revealed four levels of structural development similar to those reported by Mulligan and Mitchelmore (2009). The four levels of response were *prestructural*, *emergent structural*, *partial structural*, and *structural*.

Prestructural responses lacked any evidence of numerical or spatial structure. This strategy was very similar to the Random Arrangement strategy for the Repeating Patterns tasks. For example, Figure 4(b) shows some prestructural attempts to copy the six-dot triangular pattern in Figure 4(a). The children's responses show curved and straight lines that bear no relationship to the shape, spatial arrangement, or quantity of the given pattern.

Figure 5 shows another example, an Arrays task. The child has randomly placed counters on the cardboard frame provided (shown on right side) without observing the square structure of the array or the number of counters (shown on left side).

Prestructural responses were quite common in both groups of children at Assessment 1 but virtually disappeared after the intervention. For example, on the Triangles 1 task, 14% of the responses from IP children and 23% of those from NP children were prestructural. At Assessment 2, none of the IP children gave responses in this category, and only 8% of the NP children's responses were prestructural.

Emergent structural responses showed some features of numerical and spatial structure. Individual shapes such as squares or circles were evident. For example, it can be seen in Figure 6 that some idea of structure was beginning to emerge. The drawing did not correctly represent either the triangular shape of the array or the number of circles, but there was an apparent attempt to form rows.

Emergent structural responses were evident for both groups of children at Assessment 1 and less evident for both groups at Assessment 2. For example, 14% of the IP children gave emergent structural responses to the triangular pattern task at Assessment 1 but only 5% gave such responses at Assessment 2. By contrast, the portion of responses in this category from the NP children barely changed, from 3% to 5%.

Partial structural responses depicted several aspects of numerical and spatial structure such as triangles, grids, or arrays without being correct. Figure 7 shows a typical example. The representation clearly shows the structure of two equal rows of counters (or possibly four pairs of counters) forming a rectangular shape, but

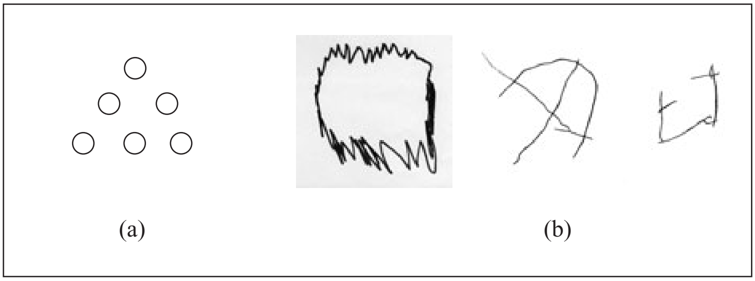

Figure 4. Triangle copying task: (a) the given pattern, and
(b) three prestructural responses.

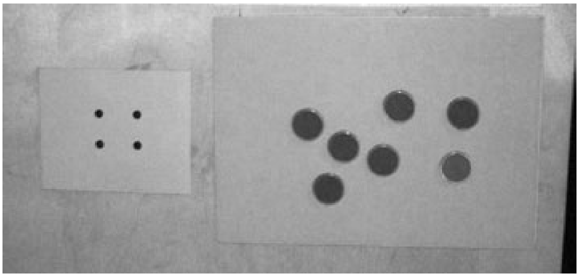

Figure 5. Prestructural response to an array-copying task.

the number of columns (or pairs) is incorrect. Nonetheless, this spatial structure appeared to reflect the child's recognition of equal groups and the possible construction of a unit of repeat in this context.

Partial structural responses were common before the intervention, comprising 32% of all responses. At Assessment 2, 31% of responses from the NP children fell into this category, but only 9% from the IP children.

Structural representations correctly integrated numerical and spatial structural features. Children often demonstrated their awareness of structure in their comments, as illustrated in the following excerpt:

[A 3-dot triangular pattern was placed in front of the child to copy with counters.]

Child: You're making numbers of patterns.
Interviewer: [Points to an inverted 3-dot triangular pattern] What about this one?
Child: Oh, that's numbers of patterns, too.
Interviewer: How?
Child: It's the same as that one [referring to the first pattern] but the other way.

At Assessment 1, 33% of all the children gave structural responses. At Assessment

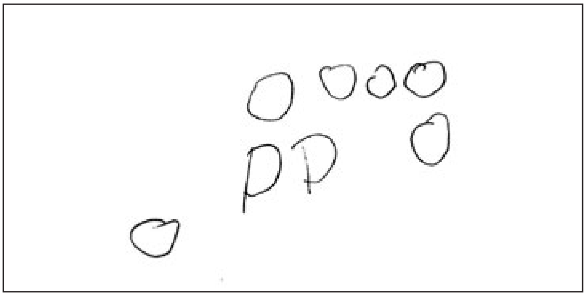

Figure 6. Emergent structural response to the task of copying the triangular array in Figure 4(a).

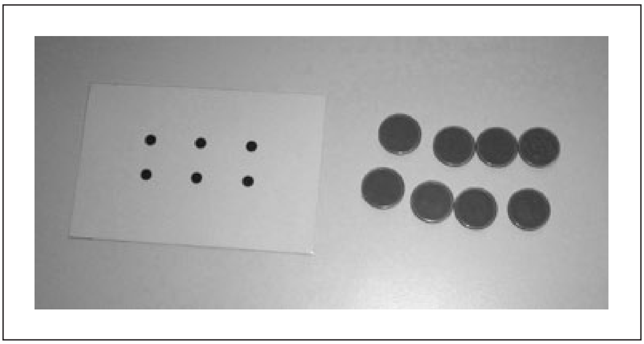

Figure 7. Partial structural response to array copying task.

2, there was very little change among the NP children (33%) but the portion of IP children giving structural responses rose to 66%.

In summary, IP children appeared to acquire a much better understanding of geometrical structure than the NP children. It was noted that children's responses to the Spatial Structure tasks were generally consistent with their responses to the Repeating Patterns tasks. In particular, children who successfully identified the unit of repeat within a variety of Repeating Patterns tasks tended to represent spatial structure patterns with correct shape, size, orientation, and spatial and numerical structure.

Growing Patterns

Growing Pattern tasks were administered only at Assessment 3, after the 1st year of formal schooling. Analysis of both correct and incorrect solution strategies revealed four main categories of response: *repeating pattern*, *incorrect shape*, *inaccurate shape*, and *correct extension*.

Repeating pattern. Many children interpreted the given pattern as a unit of repeat and simply replicated it. Although this response is technically correct, this strategy

was considered trivial because it does not take any account of the relationships within the given stimulus figure. Figure 8 shows a typical response in this category. This strategy was common among NP children (58%) but infrequent in the IP group (15%).

Incorrect shape. Some children extended the patterns using shapes that did not match the given squares or triangles and frequently had an incorrect number of elements. Figure 9 shows a typical response in this category.

Among the NP children, 33% of the responses did not correctly represent a triangular or square shape. In contrast, none of the IP children gave a response in this category.

Inaccurate shape. Some children seemed to form a holistic impression of the shape and realize that it was increasing in size but were unable to construct the correct continuation. Figure 10 shows the result of application of a strategy employing an inaccurate shape.

Only one NP child (8%) reached this level of sophistication. In contrast, nearly 30% of the responses made by the IP children fell into this category.

Correct extension. Children whose responses fell into this category indicated through their explanations that they were aware of both the numerical and spatial structure of the pattern, and most of them were able to extend the pattern indefinitely. Only the IP children continued and justified growing patterns of squares (48%) and triangles (38%) correctly.

The following excerpt demonstrates one child's conception of the square pattern as growing systematically in two dimensions.

Interviewer: Can you tell me what is happening each time we make the square bigger?

Child: Yeah, here it has one, then it has two and two lines and it's bigger. Then this one has three and three lines and then four and four lines.

Interviewer: What do you mean, four and four lines?

Child: See, there's four in each line.

Interviewer: So, what would the next one in my pattern be?

Child: Um . . . five and five lines.

This child's response illustrates how children who gave responses in this category recognized not only the spatial structure of each element (square, triangle) but also

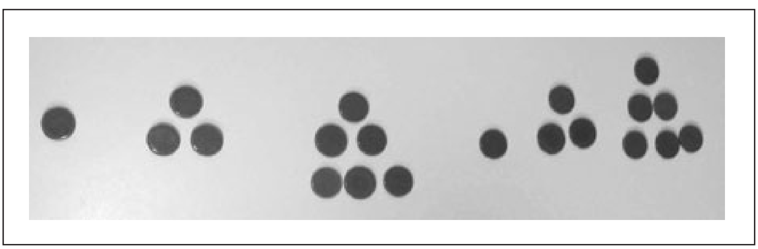

Figure 8. Triangular pattern extended by simple repetition.

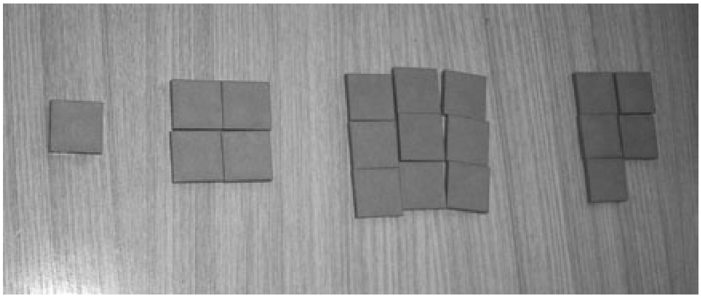

Figure 9. Squares pattern extended by a nonsquare shape.

the numerical pattern (1, 2, 3, 4, 5, . . .) relating them.

In summary, the results on the Growing Patterns tasks showed a large difference between the responses of the IP and NP children. Many IP children could extend triangular and square growing patterns successfully, and several more made only counting errors. By contrast, although a few of the NP children appeared to recognize the structure of these patterns, none extended them correctly.

DISCUSSION

Although this study was limited to two preschools, it has provided empirical evidence that young children can develop sophisticated patterning concepts and strategies prior to commencing kindergarten, and that children as young as 4 years of age can engage in prealgebraic thinking. We now examine our three research questions in turn.

Characteristics of Mathematical Patterning Among Preschool Children

Although a significant portion of the preschool children in this study initially gave apparently random responses to the various patterning tasks presented, most

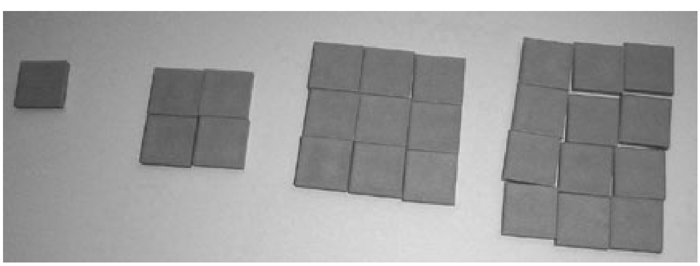

Figure 10. Squares pattern illustrated in the leftmost three figures extended in the rightmost figure by an inaccurate shape.

of them had already developed strategies that were effective in limited situations. Thus, most children could copy repeating patterns using direct comparison and continue a simple alternating pattern. However, apart from the children who experienced the patterning intervention, they were generally unable to extend more complex repeating patterns.

Over the 6-month period, all children virtually ceased giving random responses and the direct comparison strategy also became less common. We may surmise that these changes were due to the many constructive activities that took place in both preschools, in which children's attention was inevitably drawn to comparisons between their constructions and the objects to be copied.

We may characterize children's spontaneous patterning, as observed in this study, as local rather than structural. That is, their strategies were based on relations between adjacent elements rather than on the overall pattern. Regular preschool activities usually aim to heighten children's awareness of such local relations, including the alternation relation in AB repeating patterns. However, children's responses to the Spatial Structure tasks showed that most of them were only just beginning to be aware of structure in the wider sense, and then only in the simplest of situations. In particular, without the intervention, only a few children seemed to recognize the unit-of-repeat structure of repeating patterns.

Immediate Effects of the Patterning Intervention

The intervention resulted in gains in children's understanding of simple repetition and spatial patterns well beyond those made by the comparison group. On the Repeating Patterns tasks, the limited alternation strategy was virtually replaced by the much more flexible unit-of-repeat strategy and, as a result, some children were able to integrate and transfer patterns in a variety of ways. On the Spatial Structure tasks, the majority of the IP children gave correctly structured responses at the end of the intervention period. Whereas at the start of the study most of their responses did not correctly represent such elementary features as rows of counters or the overall geometrical shape, afterward they were able to show a variety of structural features (shape, size, and orientation, as well as spatial and numerical structure).

It seems that the intervention had drawn children's attention to structure at a far deeper level than is achieved through regular preschool constructive activities. A likely source of this change is the teaching strategy, adopted throughout the intervention, whereby teachers repeatedly encouraged children to look for structural similarities and differences between the given pattern and their copy of it. According to Mason, Drury, and Bills (2007), "becoming aware of similarities and differences results in stressing or fore-grounding and consequently ignoring or back-grounding" (p. 55), as a result of which children may have been able to focus on underlying concepts such as unit of repeat and array structure. We conjecture that the IP children had become aware of concepts such as collinearity and equal spacing as well as primitive generalizations such as "many patterns have a unit of repeat."

Long-Term Effects of the Intervention

The effects of the intervention were sustained 1 year after its conclusion, in that the IP children continued to outperform the NP children on the Repeating Patterns tasks. However, the most impressive result was the success of the IP children on tasks not included in the intervention: The majority of the IP children could at least make a good attempt at extending growing patterns, whereas most of the NP children simply replicated the given pattern. The IP children also outperformed the NP children on a standard numeracy assessment.

It is important to consider how the IP children may have developed the ability to extend complex growing patterns. The kindergarten teachers of both the IP and NP children all indicated that they had taught patterns according to the state curriculum, which only mentions repeating patterns that focus on alternating items and simple growing number patterns presented as skip counting (e.g., 2, 4, 6). The teachers had not mentioned growing patterns such as the ones presented at Assessment 3. So the effect seems to have been in some way the result of the intervention. It is possible that the IP children had not only become aware of many examples of numerical and spatial structure but also acquired a greater tendency to look for mathematical patterns. They would certainly have been able to recognize the structure of the individual components (squares and triangles). Analysis of the interview transcripts suggests that the IP children spontaneously generalized their idea of a pattern from one in which the successive components are identical, as in repeating patterns, to one in which there is a constant relationship between them, as in growing patterns.

It is important also to look for possible explanations of why the IP children's numeracy level was greater than the NP children's 1 year after the intervention. The participating children had been dispersed across a large number of schools, so it is unlikely that the IP children had been exposed to more effective early numeracy teaching than the NP children. We believe that the explanation is to be found in the IP children's concept of unit of repeat—the factor that most clearly differentiated them from the NP children. When a repeating pattern is analyzed in terms of its unit of repeat, the configuration is broken down into equal-sized groups. This approach allows for the development of counting techniques that are more effective than unitary counting. During the intervention, children were frequently observed counting rhythmically and, indeed, several were heard to use skip counting (e.g., "2, 4, 6") and the language of multiplication (e.g., "3 times"). This skill may have made it possible for the IP children to learn early numeracy concepts (for example, the 10s structure of the numeration system) more effectively in kindergarten. We would conjecture that the IP children would have a greater appreciation of composite units and experience a smoother transition to multiplicative thinking later.

The results of this study support the argument put forward by Mulligan and Mitchelmore (2009) that Awareness of Mathematical Pattern and Structure (AMPS) is a general feature of young children's cognition that is associated with mathematical achievement. The intervention included many activities that would have strengthened the preschool children's AMPS, and the result was a level of understanding that readily transferred to more complex patterning and counting tasks 1 year later.

The IP children's success in dealing with growing patterns may also be seen as prealgebraic thinking. This study therefore supports the findings of Blanton and Kaput (2002) and Carraher et al. (2006), who highlight the early development of quantitative relationships in emergent algebraic thinking.

CONCLUSION AND IMPLICATIONS

The present study has shown that an early patterning assessment instrument can reliably inform the development of a specific learning model. Like the Building Blocks program (Clements & Sarama, 2007b), gaining insight into children's natural mathematical thinking about patterning can guide the development of learning trajectories and a research-based curriculum (Clements, 2007). In this initial study, a preschool intervention program led to early development of the notion of unit of repeat and several elementary spatial concepts. These achievements may, in turn, facilitate the development of multiplicative and functional thinking and an improvement in Awareness of Mathematical Pattern and Structure (Mulligan & Mitchelmore, 2009). The identification and application of the unit of repeat was found to be fundamental to the children's recognition of the pattern structure. Generalizing and symbolizing pattern structure, albeit simple repetition, was critical to informing the specific learning model on patterning, and subsequently the design of an instructional framework. The other advantage was the natural link to counting patterns and the formation of the notion of equal groups (i.e., composite units), particularly in the border patterns. Moreover, the findings support Mason et al.'s (2007) assertion that young children have natural powers of generalization.

One of the important implications of this study is that very young children are underchallenged in their preschool experiences. Our findings indicate that, given opportunities to engage in mathematical experiences that promote emergent generalization, children are capable of abstracting complex patterns before they enroll in kindergarten. The crucial components are exposure to a variety of patterns in differing modes and orientations, and scaffolding by an adult to encourage justifying and transferring these patterns to other media.

Warren (2005) asserted that 9-year-old children may find growing patterns more difficult than repeating patterns because of an overemphasis on repeating patterns in early mathematics curricula. Our findings suggest that it may rather be due to the way repeating patterns are treated. Responses from the NP group, as well as the descriptors in the state syllabus (Board of Studies NSW, 2002), suggest that teachers restrict their examples to alternating patterns—which would severely reduce the likelihood of children gaining the powerful concept of unit of repeat. A simple curriculum change could have a significant impact on children's subsequent learning.

Further research with larger samples and more diverse samples, as well as other forms of media, is needed to explore the impact of early patterning activities on children's mathematical development both in preschool and in subsequent years of schooling. In particular, research is needed to determine whether an explicit focus

on patterning could later promote structural development as conjectured above—for example, in multiplicative and functional thinking—as well as in measurement and symbolization. The effect of a patterning program on visualization skills should also be investigated, as these skills also influence mathematics learning (Arcavi, 2003).

A crucial factor in implementing a more pattern-oriented curriculum will be how easily teachers can develop the pedagogical content knowledge that will enable them to incorporate concepts such as *unit of repeat* into their teaching.

Further Research

Three subsequent iterations of the refined intervention program have been extended to larger and more diverse samples in New South Wales preschool centers. This includes replication at the initial experimental preschool, and adaptations of the program in 14 preschools serving Indigenous communities in rural and regional settings. Aligned with the expansion of the project is the development of a teacher professional development program, the Patterns and Early Algebra Program (PEAP), central to improving professional knowledge and practice in early numeracy. This work has directly supported and informed Australian Government initiatives in research and curriculum in early numeracy—*Mathematical thinking of preschool children in rural and regional Australia: Research and practice* (Bobis, Papic, & Mulligan, 2010; Papic, Mulligan, & Bobis, 2010). More specifically, evidence of the relationship between patterning and structural awareness in early mathematics was seminal to the development of the broader Pattern and Structure Mathematics Awareness Program focused on students' learning trajectories in the early years of formal schooling (Mulligan & Mitchelmore, 2009; Mulligan, Mitchelmore, English, & Robertson, 2010). Together with new government-funded studies, for example, Mathematical Modelling in the Early School Years (English), Reconceptualizing Early Mathematics Learning (Mulligan & English), Patterns and Early Algebra Preschool Professional Development Project (Papic, Mulligan & Highfield), the Australian research on early mathematics learning promises to contribute to a more coherent body of evidence supporting the central role of patterning, structural relationships, and generalization in early mathematics curricula (Carraher et al., 2006; Kaput, 2008). One of the main challenges will be enabling professionals to develop and evaluate a new approach to children's mathematics learning—one that integrates patterning and structural relationships in mathematics so that a more holistic outcome may be achieved much earlier than previously expected.

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